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LETTER TO THE EDITOR

New bases of representation for the unitary parasupersymmetry algebra

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Abstract
Representation bases of unitary parasupersymmetry algebra of arbitrary order p is constructed by some one-dimensional models which are shape invariant with respect to *the main quantum number n*. Consequently, the isospectral Hamiltonians and their exact solutions are obtained as labelled by *the main quantum number n*.

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Supersymmetry, symmetry between the fermionic and bosonic degrees of freedom, is playing an important role in many recent developments in non-relativistic quantum mechanics [1–6]. Parafermi and parabose statistics are natural extensions of the usual Fermi and Bose statistics [7–9]. Fermi and Bose statistics describe the one- and two-dimensional representations of the permutation group, while the parafermi and parabose statistics describe higher dimensional representations of the same group. In fact, parasupersymmetry algebra has provided a nice symmetry between parafermions and parabosons. For the first time, Rubakov and Spiridonov introduced parasupersymmetry algebra which describes an essential symmetry between bosons and parafermions of order 2 [10]. However, later on, Khare introduced the non-unitary parasupersymmetry algebra of arbitrary order p with the parafermionic generators Q_1 and Q_2 and bosonic generator H as [11]

$$Q_1^p Q_2 + Q_1^{p-1} Q_2 Q_1 + \cdots + Q_1 Q_2 Q_1^{p-1} + Q_2 Q_1^p = 2p Q_1^{p-1} H \tag{1a}$$

$$Q_2^p Q_1 + Q_2^{p-1} Q_1 Q_2 + \cdots + Q_2 Q_1 Q_2^{p-1} + Q_1 Q_2^p = 2p Q_2^{p-1} H \tag{1b}$$

$$Q_1^{p+1} = Q_2^{p+1} = 0 \tag{1c}$$

$$[H, Q_1] = [H, Q_2] = 0. \tag{1d}$$

In the non-unitary parasupersymmetry algebra (1), the parafermionic operators Q_1 and Q_2 are not Hermitian conjugations of each other, hence relations (1) are not closed under the

Hermitian conjugation. If we choose $Q_1 = Q, Q_2 = Q^\dagger$ and $H = H^\dagger$, then we get the Khare–Rubakov–Spiridonov unitary parasupersymmetry algebra of arbitrary order p with the parasupercharges Q and $Q^\dagger$ and bosonic Hamiltonian H as follows [11, 12]:

$$Q^p Q^\dagger + Q^{p-1} Q^\dagger Q + \cdots + Q Q^\dagger Q^{p-1} + Q^\dagger Q^p = 2p Q^{p-1} H \tag{2a}$$

$$Q^{p+1} = 0 \tag{2b}$$

$$[H, Q] = 0 \tag{2c}$$

together with their Hermitian conjugations. Therefore, the relations of unitary parasupersymmetry algebra given in (2) are closed under Hermitian conjugation. At the same time, the Khare–Rubakov–Spiridonov unitary parasupersymmetry algebra has been successfully realized by many quantum solvable models [10–15]. On the other hand, the factorization method [16] has been used extensively for obtaining algebraically the exact solutions of the one-dimensional quantum models named shape-invariant potentials [12, 17–26]. Recently, most of the one-dimensional shape-invariant solvable quantum mechanical models have been classified into two bunches. The first bunch [27] includes models for which the shape invariance parameter is the main quantum number n. Furthermore, in the second bunch [28] the shape invariance parameter of the models is the secondary quantum number m. Meanwhile, it has been shown in [28] that the Khare–Rubakov–Spiridonov unitary parasupersymmetry algebra of arbitrary order p is realized by the shape-invariant quantum mechanical models so that the algebra can be represented by the quantum mechanical states of the models. For realizing the algebra, it has also been shown that the bosonic Hamiltonian involves $p + 1$ isospectrum Hamiltonians. This fact has also been studied in detail for the second bunch of the shape-invariant models in [28].

The master function $A(x)$ was introduced as a polynomial of at most degree 2, where the non-negative weight function $W(x)$ depended on the master function in the interval (a, b). The weight function $W(x)$ is determined in such a way that the expression $(A(x)W(x))'/W(x)$ becomes a polynomial of at most degree 1. Also, the interval (a, b) is chosen so that the expression $A(x)W(x)$ and its derivatives vanish at both ends. In the first class, obtained from the factorization of the Schrödinger equation with respect to *the main quantum number n*, the superpotential was explained in terms of the master function, the corresponding weight function and also *the main quantum number n* [27, 29] (*The main quantum number n* is an arbitrary non-negative integer, and also *the secondary quantum number m* is an arbitrary non-negative integer with a maximum value equal to n.) The second class was derived by factorizing the Schrödinger equation with respect to *the secondary quantum number m*, in which the superpotential was explained in terms of the master function, its weight function and also *the secondary quantum number m* [28]. Therefore, the superpotentials were labelled in terms of n and m for the first class, and in terms of m for the second class. In [28], we obtained representations for arbitrary-order unitary parasupersymmetry algebra where they were constructed by the solutions corresponding to the quantum models obtained from shape invariance on *the secondary quantum number m*. Consequently, we introduced isospectral Hamiltonians labelled by *the secondary quantum number m*.

In [27], by using shape invariance on *the main quantum number n*, we obtained solutions of the first class of exactly solvable models in terms of a multiplier of the orthogonal special functions. In this letter, for some models of the first class with corresponding non-negative weight function as

$$W(x) = A^\lambda(x) \tag{3}$$

we realize the representation of an arbitrary-order unitary parasupersymmetry algebra, and consequently, we represent the explicit form of isospectral Hamiltonians. We also calculate their spectra in terms of the parameters of master function $A(x)$ (i.e. A'', $A'(0)$ and $A(0)$) and the parameter λ. Furthermore, following the master function theory, the parameter λ and the interval (a, b) are determined so that the expression $A^{\lambda+1}(x)$ and all its derivatives vanish for the terminal points of the interval. Some special cases exist for the superpotentials of Rosen–Morse II, Eckart, and for the superpotential corresponding to the master function $A(x) = x(1-x)$ with weight function $W(x) = x^{\lambda}(1-x)^{\lambda}$ with $\lambda > -1$, which satisfy condition (3). As an example, we consider the Rosen–Morse II superpotential with the corresponding master function $A(x) = 1 - x^2$ and weight function $W(x) = (1-x)^{\alpha}(1+x)^{\beta}$. If we put $\alpha = \beta = \lambda$ with $\lambda > -1$, with of course, $-1 < x < +1$, then the weight function becomes $W(x) = (1-x^2)^{\lambda}$.

For quantum solvable models introduced in [27] with corresponding weight function satisfying condition (3), we get the factorized Schrödinger equations ($\hbar = 2M = 1, m = 0$) as

$$A^{\dagger}(n)A(n)\psi_n(\theta) = E(n)\psi_n(\theta) \qquad A(n)A^{\dagger}(n)\psi_{n-1}(\theta) = E(n)\psi_{n-1}(\theta) \tag{4}$$

with the following solutions:

$$\psi_n(\theta) = a_n \left[A^{-\lambda/2}(x) \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n A^{n+\lambda}(x) \right]_{x=x(\theta)} \tag{5}$$

where a_n is the normalization coefficient. In equation (5), the change of variable θ is obtained from the equation

$$\frac{\gamma}{\cot \gamma\theta} = \frac{A'(x)}{2} \tag{6}$$

where the constant γ is defined in terms of the master function as in

$$\gamma := \sqrt{-\frac{A'^2(0) - 2A''A(0)}{4}}. \tag{7}$$

Note that γ may be real or pure imaginary. For example, γ is equal to 1 and i if we choose the master function $A(x)$ as $x^2 + 1$ and $x^2 - 1$, respectively. Therefore, from equation (6) it is clear that x is expressed in terms of θ as a trigonometric function or a hyperbolic function depending upon which one of the values 1 or i is used for the constant γ. The explicit forms of the energy spectra of the partner Hamiltonians (4) and the raising and lowering operators $A^{\dagger}(n)$ and $A(n)$ are calculated as [12]

$$E(n) = -n(n+2\lambda)\gamma^2 \tag{8}$$

$$A^{\dagger}(n) = \frac{\mathrm{d}}{\mathrm{d}\theta} + W_n(\theta) \qquad A(n) = -\frac{\mathrm{d}}{\mathrm{d}\theta} + W_n(\theta) \tag{9}$$

where $W_n(\theta)$ is the superpotential:

$$W_n(\theta) = (n+\lambda)\frac{\gamma}{\cot \gamma\theta}. \tag{10}$$

Equations (4) describe the motion of a particle on the θ-axis in the presence of the partner potentials

$$V_{n,\pm}(\theta) = (n+\lambda)\left[\frac{(n+\lambda)\sin^2\gamma\theta \pm 1}{\cos^2\gamma\theta}\right]\gamma^2. \tag{11}$$

Equations of shape invariance (4) can be rewritten as the raising and lowering relations on the wavefunction as

$$A^{\dagger}(n)\psi_{n-1}(\theta) = \sqrt{E(n)}\psi_n(\theta) \qquad A(n)\psi_n(\theta) = \sqrt{E(n)}\psi_{n-1}(\theta). \tag{12}$$

It is obvious that shape invariance symmetry described in equations (4) concludes

$$W_n^2(\theta) + \frac{\mathrm{d}W_n(\theta)}{\mathrm{d}\theta} - E(n) = W_{n+1}^2(\theta) - \frac{\mathrm{d}W_{n+1}(\theta)}{\mathrm{d}\theta} - E(n+1). \tag{13}$$

We note that the second equation in (4) gives the first-order differential equation as

$$A(0)\psi_0(\theta) = 0 \tag{14}$$

for $n = 0$. The exact solution of equation (14), i.e. the ground state $\psi_0(\theta)$, is

$$\psi_0(\theta) = b\frac{\gamma^\lambda}{\cos^\lambda \gamma\theta} \tag{15}$$

where b is the normalization coefficient of the ground state wavefunction. Therefore, by using the first equation in (12), the algebraic solution of the Schrödinger equations (4) in terms of the ground state $\psi_0(\theta)$ becomes

$$\psi_n(\theta) = \frac{A^\dagger(n)}{\sqrt{E(n)}}\frac{A^\dagger(n-1)}{\sqrt{E(n-1)}}\cdots\frac{A^\dagger(1)}{\sqrt{E(1)}}\psi_0(\theta). \tag{16}$$

These points enable us to consider the quantum models for which we can analytically calculate the representation of Khare–Rubakov–Spiridonov unitary parasupersymmetry algebra of arbitrary order p, while *the main quantum number n* separates the isospectral Hamiltonians from each other. This is in parallel with what has been done before using *the secondary quantum number m* in [28]. Indeed, we are led to represent new bases for the arbitrary-order unitary parasupersymmetry algebra by some of the one-dimensional quantum models such as the Rosen–Morse II and Eckart superpotentials. We define the generators Q and $Q^\dagger$ and H as $(p+1)\times(p+1)$ matrices with the following matrix elements:

$$\begin{aligned}
(Q)_{nn'} &:= A(n)\delta_{n+1,n'} \\
(Q^\dagger)_{nn'} &:= A^\dagger(n')\delta_{n,n'+1} \\
(H)_{nn'} &:= H_n\delta_{n,n'} \qquad n, n' = 1, 2, \ldots, p+1
\end{aligned} \tag{17}$$

in which we use the following ansatz for the Hamiltonians H_n:

$$\begin{aligned}
H_n &= \tfrac{1}{2}A(n)A^\dagger(n) + \tfrac{1}{2}C_n \qquad n = 1, 2, \ldots, p \\
H_{p+1} &= \tfrac{1}{2}A^\dagger(p)A(p) + \tfrac{1}{2}C_p.
\end{aligned} \tag{18}$$

Then, equation (2b) and its Hermitian conjugate are automatically satisfied. With the help of the ansatz (18), equation (2c) and its Hermitian conjugate conclude

$$E(n) - E(n+1) = C_{n+1} - C_n. \tag{19}$$

Again, substituting the ansatz (18) in equation (2a) and using the shape invariance relation (4) we get

$$C_p = \frac{1}{p}\sum_{n=1}^{p} E(n) - E(p). \tag{20}$$

Thus, from relations (19) and (20), the coefficients C_n can be determined as

$$C_n = \frac{1}{p}\sum_{n'=1}^{p} E(n') - E(n) \qquad n = 1, 2, \ldots, p \tag{21}$$

where their explicit forms are calculated as

$$C_n = -\left[\frac{2p^2 - 6n^2 + 3p + 1}{6} + (p - 2n + 1)\lambda\right]\gamma^2 \qquad n = 1, 2, \ldots, p \tag{22}$$

in terms of the parameter γ. It is easy to show that the Hamiltonians $H_1, H_2, \ldots, H_{p+1}$, with the coefficients C_n as in (22), have the following eigenvalue:

$$E = \frac{-1}{2}\left[\frac{2p^2+3p+1}{6} + (p+1)\lambda\right]\gamma^2. \tag{23}$$

We see that E is independent of n. This means that the Hamiltonians $H_1, H_2, \ldots, H_{p+1}$ have the same spectrum so that they are called isospectral Hamiltonians. Here, the subscripts of the isospectral Hamiltonians $H_1, H_2, \ldots, H_{p+1}$ are just *the main quantum number n*. This is the main difference between the isospectral Hamiltonians and those introduced in [28]. Furthermore, there is a difference in the spatial functionality of the scalar potentials. Thus, one can introduce $p+1$ isospectral Hamiltonians $H_1, H_2, \ldots, H_{p+1}$ which describe the motion of a particle in the presence of quantized scalar potentials, on the θ-coordinate in terms of *the main quantum number n*:

$$\begin{aligned} H_n &= \frac{1}{2}\left[-\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + V_{n,-}(\theta) + C_n\right] \qquad n = 1, 2, \ldots, p \\ H_{p+1} &= \frac{1}{2}\left[-\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + V_{p,+}(\theta) + C_p\right] \end{aligned} \tag{24}$$

with the following eigenvalue equations:

$$H_n\psi_{n-1}(\theta) = E\psi_{n-1}(\theta) \qquad n = 1, 2, \ldots, p+1. \tag{25}$$

Now, by introducing the column matrix $\Psi(\theta)$ as a new basis with $(p+1)$ rows as

$$(\Psi(\theta))_n := \psi_n(\theta) \qquad n = 0, 1, \ldots, p \tag{26}$$

the representation of unitary parasupersymmetry algebra of order p practically realizes as

$$H\Psi(\theta) = E\Psi(\theta) \qquad Q\Psi(\theta) = \begin{pmatrix} \sqrt{E(1)}\psi_0(\theta) \\ \sqrt{E(2)}\psi_1(\theta) \\ \vdots \\ \sqrt{E(p)}\psi_{p-1}(\theta) \\ 0 \end{pmatrix} \qquad Q^\dagger\Psi(\theta) = \begin{pmatrix} 0 \\ \sqrt{E(1)}\psi_1(\theta) \\ \sqrt{E(2)}\psi_2(\theta) \\ \vdots \\ \sqrt{E(p)}\psi_p(\theta) \end{pmatrix}. \tag{27}$$

Here, the adjoint conjugate parafermionic generators Q and $Q^\dagger$ have the following explicit forms:

$$(Q)_{nn'} = \left(-\frac{\mathrm{d}}{\mathrm{d}\theta} + (n+\lambda)\frac{\gamma}{\cot\gamma\theta}\right)\delta_{n+1,n'} \qquad (Q^\dagger)_{nn'} = \left(\frac{\mathrm{d}}{\mathrm{d}\theta} + (n+\lambda)\frac{\gamma}{\cot\gamma\theta}\right)\delta_{n,n'+1}. \tag{28}$$

Therefore, with a view to the work of [28] where the representation of the Khare–Rubakov–Spiridonov unitary parasupersymmetry algebra of arbitrary order p was obtained for all quantum solvable models of shape invariant with respect to *the secondary quantum number m*, we can get the representation new bases of the Khare–Rubakov–Spiridonov unitary parasupersymmetry algebra of arbitrary order p for some of the solvable superpotentials obtained from shape invariance with respect to *the main quantum number n* which they satisfy in condition (3). These representations are realized by introducing $p+1$ isospectral Hamiltonians (24). These models, which are quantized by *the main quantum number n*, describe the motion of a particle on the θ-coordinate in the presence of their own related scalar potentials. Also, the rows of parastate $\Psi(\theta)$, which represent unitary parasupersymmetry algebra of arbitrary order p, are distinguished from each other by *the main quantum number* n. This is the reverse of the result of [28], because the rows of parastate had been labelled there in terms of *the secondary quantum number* m.

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